2022

Full Marks - 80

Time - 3 hours

The figures in the right-hand margin indicate marks

Answer *all* questions

Part-I

1. Answer the following : 1×12

a) Find supremum and infimum of

$$\left\{\frac{1}{n} : n \in N\right\}.$$

b) State Archimedean principle.

c) Give an example of a set which is bounded above but unbounded below.

d) Let F be an ordered field.

Show $0<a<b \Rightarrow 0<\frac{1}{b}<\frac{1}{a},\ a,b \in F.$

e) If $x_n = (-1)^n 2^n$, write the kinds of Divergence sequence.

f) Give an example of an unbounded sequence with no convergent sub sequence.

g) State Bolzano's Intermediate value Theorem.

h) Give an example of a continuous function which is not monotonic.

i) Give an example of a function which is continuous but not differentiable at $x = 0$.

j) State Caratheodory's Theorem.

k) State Darboux's Theorem.

l) Show $\lim_{x \to \infty} \frac{\log x}{x} = 0$.

Part-II

2. Answer any ***eight*** of the following : 2 × 8

a) Prove that there is no largest and no smallest real number.

b) Show that the set s = (0, 1) has no minimum and no maximum.

c) Show that N is unbounded above.

d) Find the range of the sequence $x_n = \frac{1}{2}\left(1+(-1)^n\right)$.

e) Give an example of a sequence which is convergent and has infinite range.

f) Give an example of an unbounded sequence with a convergent sub sequence.

g) Show $\lim_{x \to \infty} x^2 = \infty$ by using defination of limit.

h) Give an example of a function which is both monotonic and continuous.

i) Give an example of a function which is uniformly continuous but not bounded.

j) Show that the equation $10x^4 - 6x + 1 = 0$ has a root between 0 and 1.

Part-III

3. Answer any ***eight*** of the following : 3 × 8

a) Let $b \in R_+$. Then prove there exists $n \in N$ such that $\frac{1}{10^n} < b$.

b) Prove that Q is dense in R.

c) Prove that

$$\lim_n \left(n^{\frac{1}{n}}\right) = 0.$$

d) Discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{n}{3^n}$$

e) Prove $f(x) = x^2$ is uniformly continuous on $[a, b]$ but not on $[a, \infty]$, $a > 0$.

f) Show $\lim_{x \to \infty} x^{\frac{1}{x}} = 1.$

g) Show that $f(x) = 1 - |x|$ does not satisfy the condition of Rolle's theorem on [0, 2].

h) Let $f : [0, 1] \cup [2, 3] \to R$ defined by $f(x) = x$. Then prove f is continuous on the closed bounded set.

i) Discuss the convergence of the series when $x \in R$

$$1 - \frac{1}{2^x} + \frac{1}{3^x} - \frac{1}{4^x} + \ldots\ldots\ldots.$$

j) Obtain the lim sup and lim inf of $(-1)^n n + n$.

Part-IV

4. a) If $x, y \in R$ and $x > 0$, then prove there exists a positive integer n such that $n x > y$. 7

OR

b) Prove every bounded sequence has a convergent subsequence.

5. a) State and prove Cauchy (Fundamental) sequence theorem. 7

OR

b) Show that the series

$$\sum_{n=2}^{\infty} \frac{1}{n(\log n)^{\alpha}}$$

converges if $\alpha > 1$ and diverges if $\alpha \leq 1$.

6. a) Let x be closed and bounded subset of R and $f : x \rightarrow R$ be continuous. Then f attains its maximum and minimum. 7

OR

b) Let $f : [ab] \rightarrow R$ be differentiable on [a,b] and let $f'(a) \neq f'(b)$. If λ is a number between $f'(a)$ and $f'(b)$. Prove that there exists $c \in (a, b)$ such that $f'(c) = \lambda$.

7. a) State and prove Rolle's Theorem. 7

OR

b) State and prove Intermediate Value Theorem.

2022

Full Marks - 60

Time - 3 hours

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Answer *all* questions

Part-I

1. Answer the following : 1×8

a) Define the linear Differential Equation.

b) Define the Malthusian Law of population growth.

c) Give an example of a Euler Equation.

d) Write the Input-Output compartmental diagram of drug assimilation.

e) Write the differential equation of the model limited Growth with Harvesting.

f) Write the Basic Assumptions of Model of Battle.

g) Write Rutherford's statement for the radioactive decay.

h) Define the functions which are linearly dependent over a certain Internal.

Part-II

2. Answer any ***eight*** of the following : $1\frac{1}{2} \times 8$

a) Show that for any constant C, the function $y(x) = ce^x, x \in R$ is a solution of $\frac{dy}{dx} = y, x \in R$.

b) Define the first order differential equation which is Homogeneous.

c) Write the Balance equation of General Compartmental lake pollution model.

d) Write the characteristic equation of the differential equation

$$y'' - 5y' + 6y = 0$$

e) Write word equation for density dependent growth model.

f) Verify the exactness of the differential equation

$(x^2y^2 + xy + 1)dx + (x^2y^2 - xy + 1)dy = 0.$

g) Write Euler's equation.

h) Find an integrating factor for

$$(x - \ln y)\frac{dy}{dx} = -y \ln y.$$

i) To find the general solution of

$y'' + P_1 y' + P_2 y = f(x)$ the co-efficient must be____.

j) Formulate the differential equation of simple battle model.

Part-III

3. Answer any ***eight*** of the following : 2 × 8

a) Test the equation

$e^y dx + (xe^y + 2y)dy = 0$ for exactness and solve it.

b) Solve the initial value problem

$$y' - 2y = 4, \ \ y(0) = 0.$$

c) Find an expression the time 'T', the population to double in size.

d) Write balance equation in word form for the limited growth with harvesting.

e) Find the general solution of

$$y'' - 3y' - 4y = 0, \ \ y(0) = 1, \ \ y'(0) = 0.$$

f) Solve

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - 4y = 0.$$

g) Find the particular Integral of $(D-2)^2 y = e^{2x}$.

h) Solve $(D^2 + 3D - 10)\, y = 6e^{4x}$

where $D = \frac{d}{dx}$.

i) Determine possible direction of phase plane trajectories in the phase-plane.

j) Determine a component diagram and appropriate word equation for each of the two populations, the predatory and pray model.

Part-IV

4. a) Solve

$$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x} \log x .$$ 6

OR

b) Taking production rate 1, carrying capacity as 10, harvesting rate $\frac{9}{10}$ and initial population x_0, write the differential equation of the limited growth and harvesting model. Solve it and interprete the solution.

5. a) Define half life. If the half cycle is z, then find k interms of z. 6

OR

b) Solve the initial value problem

$$yy' = x^3 + \frac{y^2}{x};\ y(2) = 6.$$

6. a) Find the particular integral of

$(D^2 + 1)\, y = \text{cosec } x.$ 6

OR

b) Determine and sketch the family of phase-plane curves given by

$$\frac{dI}{ds} = -1 + \frac{r}{\beta S}$$

7. a) Find the equilibrium solutions of the differential equation of predator prey model

$$\frac{dx}{dt}\ \beta_1 x - c_1 xy,\ \frac{dy}{dt} = -\alpha_1 y + c_2\, xy.$$

where x, y denote prey and predator population. C_1 C_2 positive constant, β_1 per capita birth rate and α_1 is the per capita death rate. 6

OR

b) Solve

$$(x^2D^2 - 3xD + 4)y = 2x^2.$$

□□

2019

Full Marks - 80

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Answer *all* questions

1. a) If U and W are two subspaces of a finite dimensional vector space V, then prove that

$$\dim(U+W)=\dim(U)+\dim(W)-\dim(U\cap W)$$

8

b) IP $T_3 : V_3 \rightarrow V_2$ defined by

$T(x_1, x_2, x_3) = (x_1 - x_2, x_1 + x_3)$,

Show that T is a Linear transformation. 4

c) Check whether the following set of vectors is LI or LD.

$\{(1,-1,2,0,),(1,1,2,0),(3,0,0,1),(2,1,-1,0)\}$ 4

OR

d) State and prove Rank Nullity Theorem. 8

e) If U and W are subspaces of a vector space V, prove that $U \cap W$ is a subspace of V. 4

f) Find the co-ordinates of the vector $(-1,3,1)$ relative to the ordered basis 4

$B = \{(2,1,0), (2,1,1), (2,2,1)\}$

2. a) Determine whether the following system of Linear equations is consistent. 8

$x_1 + 2x_2 + 4x_3 + x_4 = 4$

$2x_1 - x_3 - 3x_4 = 4$

$x_1 - 2x_2 - x_3 = 0$

$3x_1 + x_2 - x_3 - 5x_4 = 5$

b) If T be a Linear map on V_3 defined by

$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$

then, find the matrix associated with T in the standard basis of V_3. 4

c) Determine the rank of the matrix

$$A = \begin{pmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{pmatrix}$$

by reducing it into row-reduced echelon form. 4

OR

d) i) Find the eigen values and eigen vectors of the matrix 4

$$\begin{pmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{pmatrix}$$

ii) Find the rank of the matrix $\begin{pmatrix} 3 & -1 & 2 \\ 0 & 1 & -3 \\ 6 & -1 & 1 \end{pmatrix}$ 4

e) IP λ is an eigen value of A prove that λ^n is an eigen value of A^n. 4

f) Reduce the matrix in to row reduced echelon form. 4

$$\begin{pmatrix} 1 & -1 & 1 \\ 3 & -1 & 2 \\ 3 & 1 & 1 \end{pmatrix}$$

3. a) Prove that if G is a finite group and H is a subgroup of G, then $0(H)$ divides $0(G)$ 8

b) If G is a group then prove that 4

i) $(a^{-1})^{-1} = a$ for every $a \in G$

ii) $(ab)^{-1} = b^{-1}a^{-1}$ for every $a, b \in G$

c) If H is a non-empty set of a group G with the property that 4

i) $a, b \in H$ implies $ab \in H$

ii) $a \in H$ implies $a^{-1} \in H$,

then prove that H is a subgroup of G.

OR

d) Prove that the subgroup N of a group G is a normal subgroup of G iff every left coset of N in G is a right coset of N in G. 8

e) Prove that N is a normal subgroup of G iff $gNg^{-1} = N$ for every $g \in G$.

f) Prove that Every permutation is the product of its cycle. 4

4. a) Prove that a finite integral domain is a field. 8

b) If R is a ring, for $a, b \in R$ prove that 4

i) $a(-b) = (-a)b = -(ab)$

ii) $(-a)(-b) = ab$

c) Prove that the ring of integers mod P, $\mathbb{Z}_p$ is a field where P is a prime. 4

OR

d) If U and V are ideals of a ring R, and $U+V=\{u+v: u\in U \text{ and } v\in V\}$, then Prove that U + V is also an ideal of R. 8

e) If F is a field prove that its only ideal are (O) and F its self. 4

f) Define ring homomorphism.

If $\mathbb{Z}(\sqrt{2})=\{m+n\sqrt{2}: m,n\in\mathbb{Z}\}$ and $Q:\mathbb{Z}(\sqrt{2})\to\mathbb{Z}(\sqrt{2})$ is a mapping defined by $Q(m+n\sqrt{2})=m-n\sqrt{2}$, show that Q is a ring homomorphism of $\mathbb{Z}\sqrt{2}$ on to $\mathbb{Z}\sqrt{2}$. 4

5. a) If f(x), g(x) are two polynomials in F[x], prove that $\deg(f(x)g(x))=\deg f(x)+\deg g(x)$. 8

b) Show that x^2+x+1 is irreducible over $\mathbb{Z}_2$, the ring of integers mod2. 4

c) State division algorithm for polynomials with example. 4

OR

d) State and prove Gauss' lemma. 8

e) State Eisenstein Criterion for irreducibility Test. Show that $3x^5 + 15x^4 - 20x^3 + 10x + 20$ is irreducible over Q. 4

f) Show that $f(x) = 2x^2 + 4$ is irreducible over Q but reducible over $\mathbb{Z}$. 4